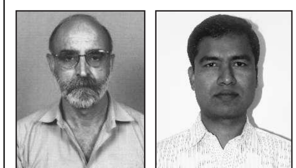
Fermi–Dirac Statistics

Derivation and Consequences

S Chaturvedi and Shyamal Biswas

After a brief exposition of the history of the Fermi–Dirac statistics, we show how this statistics emerges as a possible statistics for a quantum description of an assembly of identical and indistinguishable particles. We then present the necessary tools for computing thermodynamic properties of specific fermionic systems and highlight the role it has played in our understanding of physical phenomena ranging from transport properties of metals to those pertaining to stability of stars.



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1. Introduction

Fermi–Dirac statistics describes energy distribution in a non-(or weakly) interacting assembly of identical particles now known as fermions. Fermions are particles with 1/2-integer spin, e.g., neutrinos, electrons, quarks, protons, neutrons, ^{6}Li, ^{40}K atoms, etc., and obey the Pauli exclusion principle [1] which decrees that no more than two such particles can occupy the same quantum state. It bears the names of Enrico Fermi [2] who derived it in 1926 and of Paul Dirac [3] who derived it independently a little later in the same year. It was a very important year in the history of quantum mechanics as well as that of modern physics as it was a witness to the revolutionary transition from the old quantum theory to the new quantum theory. That electron was a spin 1/2 particle was already proposed though not well understood [4]. Nonetheless, the Pauli exclusion principle worked extremely well. It satisfactorily explained the structure of the periodic table, the fine structure in atomic spectra, anomalous Zeeman effect, Paschen–Back effect,

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Particles with 1/2-integer spin such as electrons, protons, neutrons, ^{6}Li atoms are called fermions. They obey Fermi–Dirac statistics. In contrast, those with integer spin such as photons, mesons, ^{7}Li atoms are called bosons and they obey Bose–Einstein statistics.

etc. While by 1926 one had already learnt how to quantize single-particle using the canonical commutation relations between position and momentum (first quantization) and to derive the energy spectrum of specific physical systems, the work of Fermi and Dirac laid the foundations of what is now known as second quantization – quantum mechanics of many-particle systems, where the system as a whole is required to respect a rule such as the Pauli exclusion principle. Their treatment of such systems in equilibrium at a finite temperature showed that while in the limit of high temperatures the results agree with those based on Maxwell–Boltzmann statistics, in the degeneracy limit, i.e., limit of low temperatures the results differed considerably and were in agreement with the qualitative predictions of Nernst on the 'degeneracy' of gases at low temperatures [5].

It should be mentioned, two years prior to the work of Fermi and Dirac, a similar successful attempt was made towards understanding 'degeneracy' of a gas of another type of identical particles now called bosons, e.g., photons, gluons, π^0, ^{1}H, ^{4}He, ^{7}Li, ^{14}N, ^{16}O, ^{23}Na, H_2O, ^{52}Cr, $W^\pm$ bosons, ^{87}Rb, Z bosons, Higgs bosons, etc. This was done by Einstein [6] by examining the ramifications of Bose's ideas [7] implicit in his derivation of the Planck's blackbody radiation law in the context of a gas of N identical non-interacting particles in thermal equilibrium. Spin of particles, other than that of electrons, was of course, not known at that time [8]. The theoretical connection between spin of identical particles and their statistics, Fermi–Dirac or Bose–Einstein, came to be established much later – around 1940 – in the form of the spin–statistics theorem [9]. Categorization of atoms as either bosons or fermions started around this time [10]. Later, in the late 1940s, at a more fundamental level, relativistic quantum field theory came to us as a new description of a many-particle system involving both fermions and bosons permitting interpretation of forces among fermions as exchange of bosons: exchange of photon leads to Coulomb interaction between electrons, exchange of Z boson leads to weak interaction between electrons and neutrinos, exchange of pi meson (π^0) or gluons leads to strong interaction between protons and neutrons or among quarks [11]. It is curious to note that a direct experimental evidence confirming that atoms of integral spin obey Bose–Einstein statistics appeared relatively recently in the year 1995, and that atoms of half integral spin obey Fermi–Dirac statistics, even later in the year 1999 [12].

Fermi–Dirac and Bose–Einstein statistics brought into play a new notion of purely quantum mechanical origin – the notion of indistinguishablity.

From the brief historical remarks presented above, it is evident that Fermi–Dirac statistics has played an important role in the rise of the Pauli exclusion principle from a phenomenological rule initially aimed at explaining atomic spectra to a fundamental physical principle rooted in quantum field theory. A more comprehensive account of this journey covering all related philosophical and scientific aspects can be found in [13]. Both Fermi–Dirac and Bose–Einstein statistics brought into play a new notion of purely quantum mechanical origin – the notion of indistinguishablity which sets them apart from the classical Maxwell–Boltzmann statistics. This being the case, rather than tracing the detailed historical path leading to the development of Fermi–Dirac statistics, we will focus here on the finished product and present it in a way that highlights Dirac's general perspective on statistics describing identical and indistinguishable particles within the framework of quantum statistical mechanics from which Fermi–Dirac statistics and also Bose–Einstein statistics emerge as special cases. This being done we then proceed to bring out the enormous role Fermi–Dirac statistics has played in our understanding of physical phenomena ranging from transport properties of metals to those of astrophysical importance.

Occupation number description consists in grouping together the states in the set $|i_1\rangle \otimes |i_2\rangle \otimes ...|i_N\rangle$, $i_1, i_2, ..., i_N = 1,2,..., M$ in which 1 occurs n_1 times, 2 occurs n_2 times and so on. The occupation numbers evidently add up to N and the total energy of the system corresponding to a given set of occupation numbers is given by $\sum_{i=1}^{M} n_i \varepsilon_i$.

2. Quantum Statistics Describing Identical and Indistinguishable Particles

2.1 *Identical Particles*

Consider a system consisting of N kinematically identical non-interacting particles each of which can exist in M energy states $|i\rangle, i = 1, \cdots, M$ corresponding to energies $\epsilon_1, \epsilon_2, \cdots \epsilon_M$. Let H denote the corresponding single-particle Hamiltonian. The Hilbert space $\mathcal{H}_N$ describing this composite system then consists of an N-fold tensor product of the single-particle Hilbert space $\mathcal{H}$:

$$\mathcal{H}_N = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H} \qquad (1)$$

and is of dimension M^N. The Hamiltonian for the N-particle system will have the structure:

$$H_N = H(1) + H(2) \cdots + H(N) \;, \qquad (2)$$

where $H(1)$ denotes $H \otimes I \otimes I \cdots \otimes I$ and so on. Let us choose $\{|i\rangle, i = 1, \cdots M\}$ as a basis for each $\mathcal{H}$; then the set of M^N states $|i_1\rangle \otimes |i_2\rangle \otimes \cdots |i_N\rangle, i_1, i_2, \cdots, i_N = 1, 2, \cdots, M$ serve as a basis for $\mathcal{H}_N$. We can decompose this set of M^N states by grouping together states which have the same number of 1's, 2's, $\cdots$, etc., regardless of their location in the product. Each such group is characterized by a composition of N, i.e., by a set of occupation numbers $n \equiv (n_1, n_2, \cdots, n_M)$, adding up to N. Elementary combinatorial considerations tell us that the number of states $f(n_1, \cdots, n_M)$ in each such group is given by $f(n_1, \cdots, n_M) = N!/n_1! \cdots n_M!$ and hence the canonical partition function for this system at a temperature T is given by

$$Z_N^{(\mathrm{I})}(x) = \sum_{\substack{n_i \\ \Sigma n_i = N}} \frac{N!}{n_1! \cdots n_M!} \; x_1^{n_1} x_2^{n_2} \cdots x_M^{n_M}, \qquad (3)$$

where $x \equiv x_1, \cdots, x_M$; $x_i \equiv \exp(-\epsilon_i/k_B T)$. Since $f(n_1, \cdots, n_M)$ is a symmetric function of $n_1, \cdots, n_M$, we rewrite (3) as

$$Z_N^{(\mathrm{I})}(x) = \sum_\lambda \frac{N!}{\lambda_1!, \cdots, \lambda_M!}\, m_\lambda(x). \qquad (4)$$

Here $\lambda \equiv (\lambda_1, \lambda_2, \cdots, \lambda_M)$, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_M$ is a partition of N and $m_\lambda(x)$ denotes the monomial symmetric function [14]

$$m_\lambda(x) = \sum x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_M^{\lambda_M} \qquad (5)$$

corresponding to the partition λ. (These functions, one for each partition λ, like the Schur functions $s_\lambda(x)$ to be encountered a little later, constitute a basis in the space of symmetric multivariate polynomials.) The sum in (3) can be carried out using the multinomial theorem to obtain

$$Z_N^{(\mathrm{I})}(x) = (x_1 + \cdots + x_M)^N, \qquad (6)$$

which is the well-known expression for the canonical partition function for infinite (uncorrected Maxwell–Boltzmann) statistics. That for the corrected Maxwell–Boltzmann statistics is obtained by dividing $Z_N^{(\mathrm{I})}(x)$ by $N!$.

The M^N states can also be viewed as the carrier space for an M^N-dimensional representation of the permutation group S_N whose elements P have a natural action on the basis states $|i_1\rangle \otimes |i_2\rangle \otimes \cdots |i_N\rangle$:

$$P|i_1\rangle \otimes |i_2\rangle \otimes \cdots |i_N\rangle$$
$$= |i_{P(1)}\rangle \otimes |i_{P(2)}\rangle \otimes \cdots |i_{P(N)}\rangle. \qquad (7)$$

Further, the Hamiltonian H_N by construction commutes with all elements of S_N – it is permutation symmetric. The reducible representation of S_N obtained by its action on the basis states can be decomposed into the irreducible representations of S_N which are in one-to-one correspondence with the partitions λ of N. All features of this decomposition are encapsulated in

$$Z_N^{(\mathrm{I})}(x) = \sum_\lambda n(\lambda) s_\lambda(x), \qquad (8)$$

The permutation group S_N plays a crucial role in defining the notion of indistinguishability in the context of non-relativistic quantum mechanics of an assembly of non-interacting identical particles. F–D and B–E statistics correspond respectively to the one-dimensional antisymmetric and symmetric irreducible representations of S_N.

where $n(\lambda)$ denotes the dimension of the irreducible representation λ of S_N and $s_\lambda(x)$ denote the Schur functions [14].

$$s_\lambda(x_1, \cdots, x_M) = \frac{\det(x_i^{\lambda_j+M-j})}{\det(x_i^{M-j})} \quad ; \quad 1 \leq i, j \leq M \ . \quad (9)$$

2.2 *Indistinguishabilty in the Permutation Group Sense*

The Hilbert space $\mathcal{H}_{phy}$ describing identical and indistinguishable particles is constructed out of $\mathcal{H}_N$ by (a) admitting only those operators on $\mathcal{H}_N$ which are permutation symmetric, i.e., those operators which, like H_N, treat all the factors in the tensor product democratically, and (b) identifying those states in $\mathcal{H}_N$ which have the same expectation values for all permutation symmetric operators. These assumptions, by the well-known Schur's lemma, imply that all states in $\mathcal{H}_N$ belonging to an irreducible representation λ of S_N count as one state of $\mathcal{H}_{phy}$.

2.3 *Identical and Indistinguishable Particles*

From the considerations given above, it is clear that partition function for statistics describing identical and indistinguishable particles is obtained by setting $n(\lambda) = 1$ in equation (8) [15]

$$Z_N^{(\mathrm{II})}(x) = \sum_\lambda s_\lambda(x). \quad (10)$$

Different statistics describing identical and indistinguishable particles correspond to different restrictions on λ's in this sum. Fermi and Bose statistics arise by restricting oneself to $\lambda = \{1^N\}$ and $\lambda = \{N\}$ corresponding respectively to the one-dimensional antisymmetric and symmetric representations of the permutation group. As noted earlier, particles with half-integer spin are found to obey Fermi statistics and those with integer spin, the Bose statistics. Other kinds of statistics like para-bose and para-fermi which correspond to other restrictions on λ have also been invoked in particle physics in the context of quarks.

In Chapter IX of his book, Dirac explictly constructs out of elements of S_N a maximal set of mutually commuting operators which commute with Hamiltonian H_N.

The essence of the notion of indistinguishablity as formulated above has its origin in the work of Dirac [16] who begins by examining the consequences of the commutativity of the Hamiltonian H_N (taken to be time-independent) with the elements of the permutation group P from the point of view of the dynamical conservation laws it gives rise to. The fact that $[P, H_N] = 0$ implies that an eigenstate of P corresponding to some eigenvalue evolves under H_N to a state with the same eigenvalue. Though all the P's commute with H_N, they do not necessarily commute with each other. Dirac then goes on to construct – out of the P's a maximal set of commuting operators – operators that commute with H_N as well as with each other. This set of operators turns out to be $\chi_k = \sum_{P\in\mathcal{C}_k} P,\ k = 1, \cdots m$, where $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m$ denote the conjugacy classes of S_N. The eigenvalues of this set of operators can then be used to divide the state space into mutually exclusive sectors each of which evolves into itself under H_N.

Dirac then goes on to show in his inimitable style that the number of such eigenvalue sets is precisely m, the number of classes, which in turn is the same as the number of partitions of N. This set includes (a) one in which all the eigenvalues are 1, and (b) one in which they take values ± 1 depending on whether the χ in question consists of even or odd permutations. Within each sector an eigenstate $|\chi\rangle$ corresponding to the eigenvalue set appropriate to that sector, and the states $f(P)|\chi\rangle$ obtained by applying all functions $f(P)$ of P (assuming that $f(P)|\chi\rangle \neq 0$) would correspond to the same eigenvalue set and are hence physically indistinguishable from each other and should be identified. To put it differently, the orbit of $|\chi\rangle$ under S_N counts as one state. For a given eigenvalue set for the χ's, Dirac does say that

Among all possible statistics permitted by the notion of indistinguishability in the permutation group sense, F–D and B–E statistics are the only ones for which the grand canonical partition function $Z(X)$ has the structure

$Z(X) = \prod_{i=1}^{M} z(X_i)$

where $z(X) = (1 + X)$ for F–D statistics and $z(X) = 1/(1 - X)$ for B–E statistics.

there would be a definite number of such physically indistinguishable states but stops short of giving any further details. The analysis presented here completes this detail.

2.4 *Fermi–Dirac Statistics*: *Thermodynamics of Fermions*

The canonical partition function for Fermi statistics is given by

$$Z_N^{(\mathrm{FD})}(x) = s_{1^N}(x)\,, \tag{11}$$

which on reverting back to the occupation number description reads

$$Z_N^{(\mathrm{FD})}(x) = \sum_{\substack{n_i \\ \Sigma n_i = N, n_i = 0,1}} x_1^{n_1} x_2^{n_2} \cdots x_M^{n_M}\,. \tag{12}$$

For the grand canonical partition function

$$\mathcal{Z}^{(\mathrm{FD})}(x) = \sum_N e^{\mu N} Z_N^F(x)\,, \tag{13}$$

we have

$$\mathcal{Z}^{(\mathrm{FD})}(x) \equiv \mathcal{Z}^{(\mathrm{FD})}(X) = \prod_{i=1}^{M}(1 + X_i), \qquad X_i = e^{-\beta(\epsilon_i - \mu)}. \tag{14}$$

Here μ denotes the chemical potential and $\beta = 1/k_B T$. All thermodynamic properties can be computed from the knowledge of $\mathcal{Z}^{(\mathrm{FD})}$ or equivalently from the grand potential $\Omega^{(\mathrm{FD})}$:

$$\Omega^{(\mathrm{FD})} = -\frac{1}{\beta}\log \mathcal{Z}^{(\mathrm{FD})} = -\frac{1}{\beta}\sum_i \log(1 + e^{-\beta(\epsilon_i - \mu)}). \tag{15}$$

Thus for the mean number $\overline{n}_i$ of particles occupying the energy state ϵ_i we have

$$\overline{n}_i = \frac{\partial}{\partial \epsilon_i}\Omega^{(\mathrm{FD})} = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1}. \tag{16}$$

This is the celebrated Fermi–Dirac distribution function which Fermi derived in his 1926 paper, where he specifically considered the case of harmonically bound particles, i.e., where the single particle energies ϵ_i were chosen to correspond to those of a three-dimensional harmonic oscillator.

The −1 in the denominator for the expression for $\overline{n}_i$ for B–E statstics gives rise to the possibility of Bose–Einstein condensation in bosonic systems.

This is to be contrasted with the corresponding formula for the Bose case:

$$\overline{n}_i = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}\,. \tag{17}$$

The two differ from each other only by the sign of 1 in the denominator and this seemingly minor difference makes a huge difference in the thermodynamic properties of assemblies of bosons and fermions. In the limit of large T this distinguishing feature, viz. ± 1 in the denominator can be neglected and one is led to the result appropriate to Maxwell–Boltzmann statistics.

The formulae above do not refer to spin of the fermions explicitly. To accommodate it, for instance, in the case of an assembly of electrons subject to a magnetic field, let $\epsilon_{i\uparrow}$ and $\epsilon_{i\downarrow}$ denote the single-particle energies parallel and antiparallel to the magnetic field then (14) and (15) read

$$\begin{aligned} \mathcal{Z}^{(\mathrm{FD})}(X) &= \prod_{i=1}^{M}(1+X_{i\uparrow})(1+X_{i\downarrow}); \\ X_{i\uparrow} &= e^{-\beta(\epsilon_{i\uparrow}-\mu)},\ X_{i\downarrow} = e^{-\beta(\epsilon_{i\downarrow}-\mu)}. \end{aligned} \tag{18}$$

$$\Omega^{(\mathrm{FD})} = -\frac{1}{\beta}\log \mathcal{Z}^{(\mathrm{FD})} = -\frac{1}{\beta}\sum_i \Big(\log(1+e^{-\beta(\epsilon_{i\uparrow}-\mu)}) + \log(1+e^{-\beta(\epsilon_{i\downarrow}-\mu)})\Big)\,. \tag{19}$$

In the absence of external magnetic fields, $\epsilon_{i\uparrow} = \epsilon_{i\downarrow} \equiv \epsilon_i$ and the expression for the grand potential acquires a factor of 2

$$\Omega^{(\mathrm{FD})} = -\frac{1}{\beta}\log \mathcal{Z}^{(\mathrm{FD})} = -\frac{2}{\beta}\sum_i \log(1+e^{-\beta(\epsilon_i-\mu)})\,. \tag{20}$$

All thermodynamic quantities can be computed from the grand potential once the single-particle energies specific to the system under consideration are given. Thus for a gas of electrons in a volume V, we have:

1. Average number of particles:

$$\overline{N} = 2\sum_i \overline{n}_i\,. \tag{21}$$

   For a given $\overline{N}$ this equation is to be used to determine μ.

2. Average energy:

$$E = 2\sum_i \epsilon_i \overline{n}_i\,. \tag{22}$$

3. Average energy per particle: $U = E/\overline{N}$.

4. Equation of state:

$$PV = -\Omega^{(\mathrm{FD})} = \frac{2}{\beta}\sum_i \log(1-\overline{n}_i)\,, \tag{23}$$

   where P denotes the pressure.

It proves convenient to rewrite (20) and (21) as

$$\Omega^{(\mathrm{FD})} = -\frac{2}{\beta}\int_0^\infty d\epsilon D(\epsilon)\log(1-\overline{n}(\epsilon))\,, \tag{24}$$

$$\overline{N} = 2\int_0^\infty d\epsilon D(\epsilon)\overline{n}(\epsilon)\,, \tag{25}$$

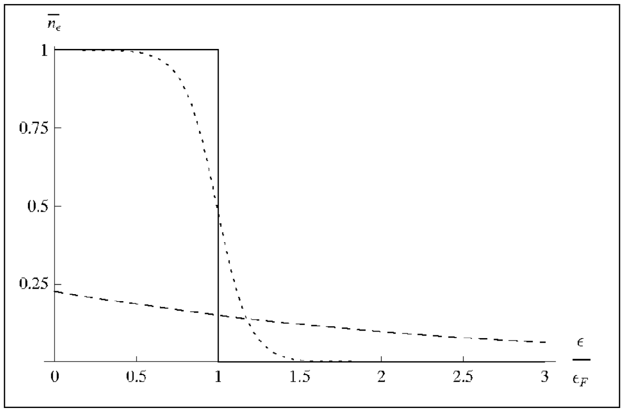

Figure 1. Solid, dotted and dashed lines represent Fermi–Dirac distribution for ideal homogeneous Fermi gas for $k_B T/\varepsilon_F = 0$, 0.1 and 2 respectively.

with $D(\epsilon) = \sum \delta(\epsilon - \epsilon_i)$ and

$$\overline{n}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}\,. \qquad (26)$$

With $D(\epsilon)$ interpreted as density of states, these formulae provide smooth passage to situations where the single-particle energy takes continuous values. As a function of ϵ, as shown in *Figure* 1, $\overline{n}(\epsilon)$ has the property that at $T = 0$, it equals 1 for $\epsilon < \mu$ and equals 0 for $\epsilon > \mu$. Fermi energy is defined as the value of ϵ beyond which $\overline{n}(\epsilon)$ vanishes, i.e., the energy $\epsilon_{F'}$ of the highest occupied level. In view of this we find that at $T = 0$, the chemical potential equals ϵ_F. At nonzero temperatures, μ is to be computed from (25).

3. Applications

We now consider a few illustrative applications of Fermi–Dirac statistics. As is clear from the discussion above, to compute the thermodynamic properties of the fermionic systems all that one needs is the expression for the single-particle energy and the expression for $D(\epsilon)$, the density of states. However, in a given situation, the integrals that appear may become quite involved and one has to resort to approximation methods valid in low or high temperature regimes. In the applications listed below some are discussed in sufficient detail while for the others we only summarize the principal results.

3.1 *Metals*: *Electron Gas Model*

The behaviour of simple (monovalent) metals can be adequately described by the electron gas model which views metals as a collection of free electrons confined to a three-dimensional box of volume V in the presence of a neutralizing positive background. Here, the single-particle energies are modelled after those appropriate to a quantum mechanical particle in a periodic box of side length L:

$$\epsilon(\mathbf{k}) = \frac{\hbar^2|\mathbf{k}^2|}{2m}; \quad \mathbf{k} = \frac{2\pi}{L}(\nu_1, \nu_2, , \nu_3); \quad \nu_i = 0, \pm 1, \pm 2 \cdots . \tag{27}$$

The formula for $D(\epsilon)$ corresponding to these single-particle energies turns out to be

$$D(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\epsilon}\,. \tag{28}$$

Substituting these in (23) and (25), one can derive the expression for the pressure P as a function of temperature and the number density $\overline{N}/V$:

$$P = \frac{\overline{N}}{V}k_B T\frac{f_{5/2}(z)}{f_{3/2}(z)}\,, \tag{29}$$

where $z = e^{\mu/k_B T}$ is the fugacity and $f_j(z)$, called the Fermi function, is given by

$$f_j(z) = \frac{1}{\Gamma(j)}\int_0^\infty dx\frac{x^{z-1}}{e^x/z+1} = \sum_{s-0}^{\infty}(-1)^{s+1}\frac{z^s}{s^r}\,. \tag{30}$$

From these calculations one finds as $T \to 0$, the pressure of the free Fermi gas, for a fixed number density, approaches a nonzero constant $\left(\frac{2}{5}\epsilon_F\left(\frac{\overline{N}}{V}\right); \ \epsilon_F = \frac{\hbar^2}{2m}\right.$ $\left(6\pi^2\left(\frac{\overline{N}}{V}\right)^{2/3}\right)$ in contrast to the Maxwell–Boltzmann statistics where it tends to 0 [17]. This nonzero pressure is called the Pauli pressure which arises solely as a consequence of Pauli exclusion principle. Bulk modulus of a metal mostly is a result of Pauli pressure. Pauli pressure also resists the gravitational collapse of a (lighter!) white dwarf or a neutron star.

For the average energy per particle, a similar calculation yields

$$U = \frac{3k_BT}{2}\frac{f_{5/2}(z)}{f_{3/2}(z)}. \qquad (31)$$

From this expression it follows that in the low temperature regime ($k_BT/\epsilon_F \ll 1$), the specific heat per particle ($C_v = \frac{\partial U}{\partial T}|_{V,N}$) for this system is given by $C_v = k_B\frac{\pi^2}{2}\left(\frac{k_BT}{\epsilon_F}\right)^1 + O\left(\frac{k_BT}{\epsilon_F}\right)^3$ [17] which is significantly different from the classical result ($3k_B/2$), and is compatible with the Nernst's heat theorem (hypothesis) [18] and matches favourably with the experimental data ([19, 20]). This result confirmed the validity of the Fermi–Dirac statistics over the Maxwell–Boltzmann statistics in the low temperature regime. Further, this expression for the specific heat, unlike its classical counterpart, also leads to a physically acceptable value for the Lorenz number for a metal as $L = \frac{K}{\sigma T} = \frac{\pi^2}{3}\left(\frac{k_B}{e}\right)^2 \approx 2.45 \times 10^{-8}$ watt-ohm/deg^2, where K is the thermal conductivity and σ is the electrical conductivity [21].

3.2 *Astrophysics*: *White Dwarfs*

Historically, one of the first applications of Fermi–Dirac distribution came from Fowler in the context of astrophysics. In late 1926, he proposed that the relationship among the density, energy and temperature of a white dwarf star could be explained by viewing it as an ideal nonrelativistic gas of electrons and nuclei which obey the Fermi–Dirac statistics [22]. He also predicted the ultimate fate of a white dwarf star, regarded as a Fermi gas, that follows from considerations of the inward gravitational pressure due to heavier nuclei and the outward Pauli pressure due to lighter electrons. This Fermi gas

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model was then used by Frenkel, Anderson and Stoner in 1928–1930 to calculate the relationship among the mass, radius, and density of the white dwarf stars, assuming them to be essentially homogeneous spheres of electron gas. Soon after this, Chandrasekhar obtained a value of critical mass in terms of solar mass (M_{O}) as

$$M_C \approx 1.4 M_{\text{O}} \tag{32}$$

for the stability of the white dwarf star applying Fermi–Dirac statistics to an ideal inhomogeneous gas of relativistic electrons [23]. Around this time, applying Fermi–Dirac statistics (possibly unaware of the previous works of Chandrasekhar and others), Landau obtained the value of critical mass not only for the white dwarf stars but also for the 'neutron' stars (even before the discovery of neutron by Chadwick).

3.3 *Nuclear Physics*

Thomas–Fermi model in nuclear physics which nowadays is commonly used as an approximation technique (even for Bose systems) was proposed as an application of Fermi–Dirac statistics in 1927. Around 1934 Fermi gas nuclear model was proposed by Majorana and Weizsacker as a similar application for calculating the binding energy of the nucleons in the nucleus.

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3.4 *Magnetism and Other Aspects of Solid State Physics*

In 1927, applying Fermi Dirac statistics for the conduction electrons, Pauli explained weak temperature dependence of the paramagnetic susceptibility

$$\chi = \frac{3}{2}\frac{\mu_B^3 \bar{n}}{\epsilon_F} + O(B^2) \tag{33}$$

of metals exposed to a weak magnetic field ($\mathbf{B} = B\hat{k}$) [24]. This explanation was compatible (to a certain extent) with the existing experimental observations, and confirmed that, (conduction) electrons obey Fermi–Dirac statistics, and that each electron behaves as a tiny (spin) magnet with two-fold degeneracy (in absence of magnetic field). Such an application modernized the theory of metals from Drude–Lorentz classical model to Sommerfeld–Bloch semiclassical model [17] around 1929. Soon after this, a diamagnetic and oscillatory effect on the magnetization of conduction electrons was added by Landau to the paramagnetic contribution obtained by Pauli to explain experimental observations on magnetization of metals particularly in strong magnetic fields and this opened the way for the theory of quantum Hall effect. In 1931, electronic band structure came as a very important step towards understanding metallic, semiconductivity and insulator behaviour of a crystalline solid body. Around this time, as a consequence of Fermi–Dirac statistics, existence of 'holes' was proposed by Heisenberg and that of positrons by Dirac. Like electrons and positrons, 'holes' also obey Fermi–Dirac statistics. This way Fermi–Dirac statistics found an area of enormous applicability in the theory of metals and semiconductors [21].

4. Later Developments

Spin-statistics theorem was proposed by Fierz and Pauli around 1940 to connect spin of particles to either Bose–Einstein statistics or Fermi–Dirac statistics [9]. This theorem led to the extension and generalization of Dirac's canonical quantization technique for photons to open quantum field theories for system of identical particles around late 1940s [11]. Subsequently, quantum hydrodynamic theory (for Bose liquid), quantum electrodynamics, Landau's theory of a Fermi liquid, BCS theory on superconductivity, Abrikosov flux lattice, Anderson localization, standard model, asymptotically free gauge theory, etc., came to us as a direct or indirect application of Fermi–Dirac statistics and revolutionized our theoretical understanding of physics. From the 1950s,

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Fermi–Dirac statistics has played a major role in the development of quantum field theory for many-particle systems which has now become the basic language for describing condensed matter and particle physics. This statistics has found numerous applications in statistical mechanics, low temperature physics, nuclear physics, semiconductor physics, low-dimensional physics, plasma physics, astrophysics, supersymmetric theory, grand unified theory, string theory, carbon nanotube physics, mesoscopic physics, ultra-cold atom physics, graphene physics, topological insulator physics, etc. [25].

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Suggested Reading

[1] W Pauli, *Z. Phys.*, Vol.31, p.765, 1925.

[2] E Fermi, Rend. Lincei, Vol.3, 145, 1926 [in Italian]; *Z. Physik*, Vol.36, p.902, 1926.

[3] P A M Dirac, *Proc. R. Soc. Lond. Ser.* A, Vol.112, p.661, 1926.

[4] W Pauli, *Z. Phys.*, Vol.31, p.373, 1925;
G E Uhlenbeck and S A Goudsmit, *Naturwiss*, Vol.13, p.953, 1925 (in German);
G E Uhlenbeck and S Goudsmit, *Nature,* Vol.117, p.264, 1926;
L H Thomas, *Nature*, Vol.117, p.514, 1926;
W Pauli, *Z. Physik.*, Vol.43, p.601, 1927.

[5] W Nernst, *Z Elektrochemie*, Vol.20, p.357, 1914.

[6] A Einstein, *Sitzungsber. Preuss. Akad. Wiss., Phys. Math.* Kl. Bericht, Vol.22, p.261, 1924; Vol.1, p.3, 1925; Vol.3, p.18, 1925.

[7] S N Bose, *Z. Phys*, Vol.26, p.178, 1924.

[8] Spin of photon was observed to be 1 in 1931 by C V Raman and S Bhagavantam, *Indian J. Phys*, Vol.6, 353, 1931; *Nature,* Vol.129, p.22, 1932.

[9] M Fierz, *Helvetica Physica Acta*, Vol.12, p.3, 1939 [in German];
W Pauli, *Phys. Rev*, Vol.58, p.716, 1940.

[10] F London, 141, 643, 1938; Vol.54, p.947, 1938.
[11] For a nice review on the same, see A Zee, *Quantum Field Theory in a Nutshell*, 2nd Ed., Princeton University Press, Princeton, 2010.
[12] B DeMarco and D S Jin, *Science*, Vol.285, p.1703, 1999.
[13] M Massimi, *Pauli's exclusion principle: the origin and validation of a scientific principle*, Cambridge University Press, Cambridge, 2005.
[14] I G Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon, Oxford, 1995.
[15] S Chaturvedi, *Phys. Rev. E*, Vol.54, p.1378, 1996;
A P Polychronakos, *Phys. Lett. B*, Vol.365, p.202, 1996;
S Chaturvedi and V Srinivasan, *Phys. Lett A,* Vol.224, p.249, 1997.
[16] P A M Dirac, *The Principles of Quantum Mechanics*, Chapter IX, Oxford University Press, Oxford, 1967.
[17] A Sommerfeld, *Z. Physik*., Vol.47, p.1, 1928; F Bloch, *Z. Physik*., Vol.52, p.555, 1929.
[18] W Nernst, *Nachr. Kgl. Ges. d. Wiss*., Göttingen, Vol.6, p.1, 1906.
[19] J A Kok and W H Keesom, *Physica*, Vol.3, p.1035, 1936;
L M Roberts, *Proc. Phys. Soc. Lond. B*., Vol.70, p.744, 1957;
W H Lien and N E Phillips, *Phys. Rev*., Vol.133, A1370, 1964; see other related (older!) articles of Keesom *et al,* e.g., *Commun. Univ. Leiden*., No.219(d), 1932; No.232(d), 1933, etc.
[20] The first mathematical calculation towards the justification of Nernst's hypothesis (which later was extended as the third law of thermodynamics) was presented by A Einstein, *Ann. Physik.,* Vol.22, p.180, 1907.
[21] N W Ashcroft and N D Mermin, *Solid state physics*, Harcourt, 1976; C Kittel, *Introduction to solid state physics*, 7th Ed., John Wiley & Sons, 1996.
[22] R H Fowler, *Mon. Not. R. Astron. Soc*., Vol.87, p.114, 1926.
[23] S Chandrasekhar, *Astrophys. J*., Vol.74, p.81, 1931; *Mon. Not. R. Astron. Soc*., Vol.95, p.207, 1935.
[24] W Pauli, *Z. Physik*, Vol.41, p.81, 1927.
[25] For a recent review on applications, particularly in atomic and condensed matter physics, see S Giorgini, L P Pitaevskii, and S Stringari, *Rev. Mod. Phys*., Vol.80, p.1215, 2008.

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